

# A note on the eight tetrahedron equations

Jarmo Hietarinta\*

Department of Physics, University of Turku

FIN-20014 Turku, Finland

and

Frank Nijhoff†

Department of Applied Mathematical Studies

University of Leeds, Leeds LS2 9JT, UK

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## Abstract

In this paper we derive from arguments of string scattering a set of eight tetrahedron equations, with different index orderings. It is argued that this system of equations is the proper system that represents integrable structures in three dimensions generalising the Yang-Baxter equation. Under additional restrictions this system reduces to the usual tetrahedron equation in the vertex form. Most known solutions fall under this class, but it is by no means necessary. Comparison is made with the work on braided monoidal 2-categories also leading to eight tetrahedron equations.

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\*E-mail: hietarin@newton.tfy.utu.fi

†E-mail: frank@amsta.leeds.ac.uk

# 1 Introduction

An important current problem in the study of integrable systems is to make an extension to higher dimensions. For 1+1 dimensions there are several well established approaches to integrability and many beautiful results have been obtained; much less is known about integrability of three dimensional systems. One of the most important approaches to 1+1 dimensional integrability is based on the Yang-Baxter equations [1], the corresponding 2+1 dimensional “tetrahedron” equations were introduced by Zamolodchikov already in the 1980 [2, 3, 4]. These equations have been under intense study during the last few years [5, 6, 7, 8, 9, 10, 11, 12, 13, 14], but many fundamental questions still remain open.

One difference between 1+1 and 2+1 dimensional integrability stems from the fact that there is no natural ordering in the two dimensional space. The Yang-Baxter equation can be derived, e.g., from the condition of factorizable scattering of point particles on a line [1] and since one can introduce a good ordering on a line there is no ambiguity in writing down the Yang-Baxter equations. Zamolodchikov’s tetrahedron equations can be derived from the conditions of factorizable scattering of straight strings [2, 12] (or particles at the intersections of strings [12]) on a plane. In the particle interpretation the scattering matrix depends on three incoming and three outgoing particles, but since there is no obvious way of defining an order in two dimensions, there is no single ordering in which the indices of the corresponding scattering matrix should be written.

Let first recall how the tetrahedron equation arises when we consider the scattering of straight strings. The basic scattering process is that of three straight strings, and if we are only interested in the particles at the intersection of the strings (the “vertex” formulation) the scattering matrix is written as  $S_{j_1 j_2 j_3}^{k_1 k_2 k_3}$  where the  $j$ ’s give the states of incoming particles and the  $k$ ’ the states of the outgoing ones. The tetrahedron equations arise when we consider the scattering of four strings [2, 12], which generically have six intersections, see Figure 1. The initial configuration looks like an arrow, and if we go to frame where the arrowhead (particle 4) is stationary, the dynamics is described fully by the way the intersection point 3 moves. Depending on the relative initial positions of the two strings at the bottom of the figure, particle 3 will pass particle 4 on the left or on the right. In each alternative there will be four basic scattering processes, in each of which a triangle will be turned over. These two alternatives should give the same result, and this condition yields the tetrahedron equations:

$$S_{j_1 j_2 j_3}^{k_1 k_2 k_3} S_{k_1 j_4 j_5}^{l_1 k_4 k_5} S_{k_2 k_4 j_6}^{l_2 l_4 k_6} S_{k_3 k_5 k_6}^{l_3 l_5 l_6} = S_{j_3 j_5 j_6}^{k_3 k_5 k_6} S_{j_2 j_4 k_6}^{k_2 k_4 l_6} S_{j_1 k_4 k_5}^{k_2 l_4 l_5} S_{k_1 k_2 k_3}^{l_1 l_2 l_3}. \quad (1)$$

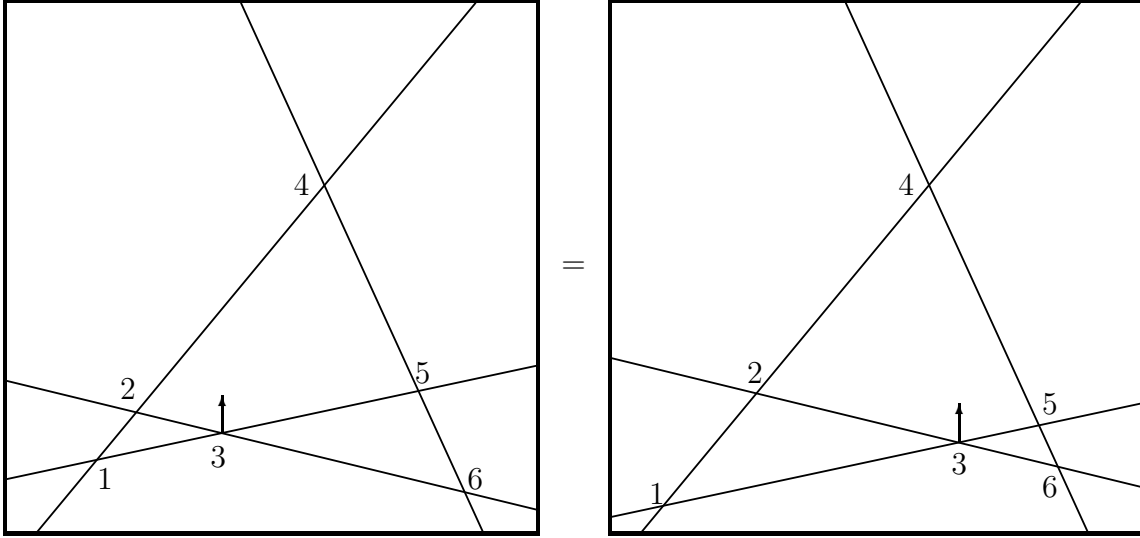


Figure 1: The starting configuration for four string (six particle) scattering. The resulting total scattering matrix should be the same for the two alternatives that differ only in the relative position of the two strings at the bottom.

Here we have used Einstein summation convention for the repeated  $k$  indices.

In writing down the above equation we have used a particular convention for the index ordering: for each triangle that turns over we have taken the indices from left to right. Since the four string configuration of Fig. 1 is not rotationally invariant “left” can always be defined, but any such ordering gives problems already when one considers the scattering of five strings.

A typical starting configuration of a five string scattering is given in Figure 2. Let us assume that in the next scattering process the triangle 123 will be turned over. In which order should we now write the indices of the corresponding scattering matrix? If we consider the triangle 123 as a part of arrow 1463 we should use  $S_{123}$ , according to the above convention, but if we consider arrow 3702 and look at the picture from right, we should use  $S_{312}$ . This problem was recognized in [2] and was taken care of by requiring that the  $S$ -matrix is invariant under cyclic index permutation, see. Eq. (3.5) of [2].

In this paper we show that this ordering ambiguity means that there are, in fact, eight tetrahedron equations (obtained from the standard one by certain index permutations) that must be satisfied simultaneously by the tetrahedron  $S$ -matrix. In Sec. 2 we give an algebraic derivation of these equations using the “obstruction” method, cf. [6, 7, 10, 14]. In Sec. 3 we give several interpretations to these equations and discuss the conditions under which these equations collapse into one. In Sec. 4 we will make a connection with the

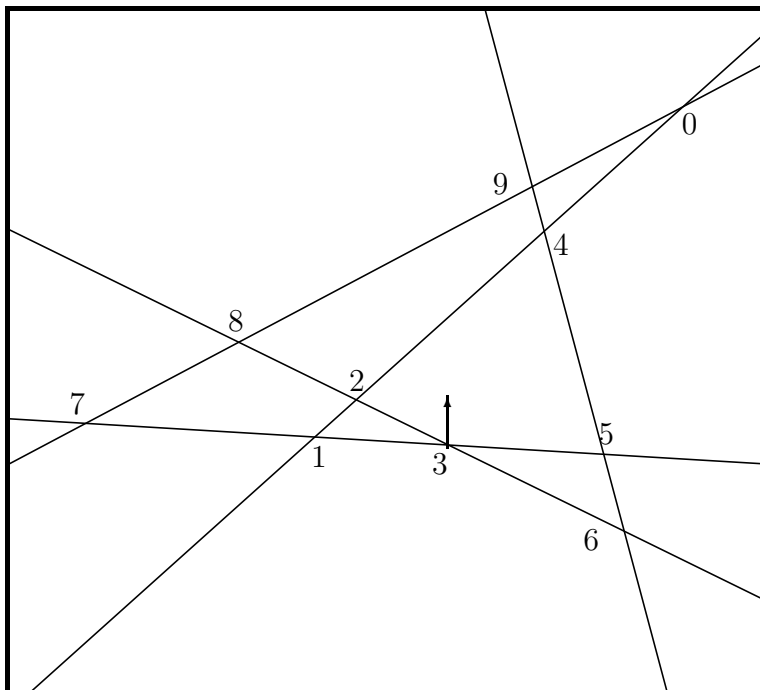


Figure 2: A typical situation with five string scattering. The first triangle to turn over is 123 of the arrow 1463, but if it is considered as part of the arrow 3702 then the corresponding scattering matrix should be labeled as 312.

notion of higher Bruhat orders introduced by Manin and Schechtman in [15, 16]. Another formulation of the tetrahedron equations is in terms of braided monoidal bicategories, cf. [17, 18], and provides an alternative way of obtaining the system of eight tetrahedron equations [19]. However, we believe our derivation is closer to the physical interpretation in terms of string scattering, furthermore we will not need to use the language of bicategories. Since our derivation relies on the obstruction mechanism, [6, 7], we hope that eventually this point of view leads to the derivation (in the spirit of [10, 11]) of explicit solutions of the system in the cases when it does not collapse to a single tetrahedron equation.

## 2 Derivation

### 2.1 Derivation of the Yang-Baxter equation

We will start by recalling the algebraic derivation of the YBE. Let us assume that we have a set of  $d$   $n \times n$  matrices which also depend on a continuous ‘spectral’ (or ‘color’)

parameters:  $\mathcal{M} = \{ {}_iM(\lambda) \in \text{End}(V_\emptyset) | i = 1, \dots, d, \lambda \in \mathbf{CP} \}$ , using the convention that the matrix indices are written on the right and the other indices on the left. For later purposes it is useful to think of the spectral parameter  $\lambda$  as being some projective vector over  $\mathbf{C}$ . Let us now assume that the matrices of  $\mathcal{M}$  do not quite commute but that their commutation is “obstructed” by some numerical coefficients  $R$ :

$${}_{j_1}M(\lambda_1)_\alpha^\beta {}_{j_2}M(\lambda_2)_\beta^\gamma = R(\lambda_1, \lambda_2)_{j_1 j_2}^{k_1 k_2} {}_{k_2}M(\lambda_2)_\alpha^\beta {}_{k_1}M(\lambda_1)_\beta^\gamma. \quad (2)$$

The obstruction coefficients  $R$  can be put into a  $d^2 \times d^2$  matrix and we can say that it operates on the product of vector spaces  $V_1 \otimes V_2$ , whose basis is given by the matrices  ${}_iM$ , themselves operating on some other vector space  $V_\emptyset$ . [This hierarchical structure will be taken one step further in the next section.] We can now use a shorthand notation and write down only the names of the vector spaces where the operation takes place

$${}_1M_\emptyset \cdot {}_2M_\emptyset = R_{12} {}_2M_\emptyset \cdot {}_1M_\emptyset. \quad (3)$$

It should be remembered that *with each vector space comes its own spectral parameter* (the parameter associated with  $V_\emptyset$  is global).

[There is an alternative way of obstructing commutativity by

$$R_{ik}^{pq}(\lambda, \mu) {}_p^mT(\lambda) \cdot {}_q^nT(\mu) = {}_k^rT(\mu) \cdot {}_i^sT(\lambda) R_{rs}^{mn}(\mu, \lambda),$$

where the  ${}_i^jT$ 's are some non-commuting quantities, each of which can be represented by a matrix acting on some vector space. Multiplying by  $R^{-1}$  from the left we can write this in the form (2), but with double indices.]

If the reversal (2) is done twice we get

$$[\delta_1 \delta_2 - R_{12} R_{21}] {}_1M_\emptyset \cdot {}_2M_\emptyset = 0,$$

which is usually taken in the strong form as the ‘unitarity’ condition

$$R_{12} R_{21} = \delta_1 \delta_2. \quad (4)$$

Taking into account the associativity of the matrix product we see that the obstruction to commutativity (3) leads to two different ways of inverting the triple  $ABC$ , namely on the one hand:  $(AB)C \rightarrow B(AC) \rightarrow (BC)A \rightarrow CBA$ , and on the other hand  $A(BC) \rightarrow (AC)B \rightarrow C(AB) \rightarrow CBA$ . Equating the two expressions obtained by elaborating these two ways, namely

$$\begin{aligned} {}_1M_\emptyset \cdot {}_2M_\emptyset \cdot {}_3M_\emptyset &= R_{12} {}_2M_\emptyset \cdot {}_1M_\emptyset \cdot {}_3M_\emptyset \\ &= R_{12} R_{13} {}_2M_\emptyset \cdot {}_3M_\emptyset \cdot {}_1M_\emptyset \\ &= R_{12} R_{13} R_{23} {}_3M_\emptyset \cdot {}_2M_\emptyset \cdot {}_1M_\emptyset, \end{aligned}$$

and

$$\begin{aligned}
{}_1M_\emptyset \cdot {}_2M_\emptyset \cdot {}_3M_\emptyset &= R_{23} {}_1M_\emptyset \cdot {}_3M_\emptyset \cdot {}_2M_\emptyset \\
&= R_{23} R_{13} {}_3M_\emptyset \cdot {}_1M_\emptyset \cdot {}_2M_\emptyset \\
&= R_{23} R_{13} R_{12} {}_3M_\emptyset \cdot {}_2M_\emptyset \cdot {}_1M_\emptyset.
\end{aligned}$$

we obtain in the strong sense the quantum Yang-Baxter equation as a condition on  $R$ :

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \quad (5)$$

which is short-hand for

$$R(\lambda_1, \lambda_2)_{j_1 j_2}^{k_1 k_2} R(\lambda_1, \lambda_3)_{k_1 j_3}^{l_1 k_3} R(\lambda_2, \lambda_3)_{k_2 k_3}^{l_2 l_3} = R(\lambda_2, \lambda_3)_{j_2 j_3}^{k_2 k_3} R(\lambda_1, \lambda_3)_{j_1 k_3}^{k_1 l_3} R(\lambda_1, \lambda_2)_{k_1 k_2}^{l_1 l_2}. \quad (6)$$

## 2.2 Derivation of the tetrahedron equation

We will next derive in a similar way the tetrahedron equation [4, 5]. We start with an indexed set of matrices  $R$  operating on a product of two vector spaces, i.e.  $\mathcal{R} = \{\alpha_{ij} R_{ij}(\lambda_i, \lambda_j) \in \text{End}(V_i, V_j) | i, j = 1, \dots, n, \alpha_{ij} = 1, \dots, m, \lambda_i \in \mathbf{CP}^2\}$ , where the spectral parameter is now a projective 3-dimensional vector. As an extension of the previous case, we assume that the  $R$ 's do not quite satisfy the Yang-Baxter equation, but rather obey

$$\begin{aligned}
&\alpha_{12} R_{12}(\lambda_1, \lambda_2) \cdot \alpha_{13} R_{13}(\lambda_1, \lambda_3) \cdot \alpha_{23} R_{23}(\lambda_2, \lambda_3) = \\
&S(\lambda_1, \lambda_2, \lambda_3)_{\alpha_{12} \alpha_{13} \alpha_{23}}^{\beta_{12} \beta_{13} \beta_{23}} \beta_{23} R_{23}(\lambda_2, \lambda_3) \cdot \beta_{13} R_{13}(\lambda_1, \lambda_3) \cdot \beta_{12} R_{12}(\lambda_1, \lambda_2), \quad (7)
\end{aligned}$$

which defines the obstruction matrix  $S$ , operating on the product of three vector spaces  $V_{(12)} \otimes V_{(13)} \otimes V_{(23)}$ , labeled now by *pairs* of integers. In (7) the internal indices of the  $R$ 's have been indicated only by the vector spaces on which they act, and there is a distributed matrix product just like in the Yang-Baxter equation over them. The external indices  $\alpha_{ij}$ ,  $\beta_{ij}$  are written out explicitly, and there is a summation over the  $\beta_{ij}$ 's.

Korepanov has successfully used (7) in constructing solutions to the tetrahedron equation [10, 11], by choosing suitably deformed solutions of the Yang-Baxter equation, see also [14].

Since the left and right hand sides of (6,7) have different index distributions (observe the positions of the repeated indices in the  $R$ 's) we will also need another reversal

$$\begin{aligned}
&\alpha_{23} R_{23}(\lambda_2, \lambda_3) \cdot \alpha_{13} R_{13}(\lambda_1, \lambda_3) \cdot \alpha_{12} R_{12}(\lambda_1, \lambda_2) = \\
&\tilde{S}(\lambda_3, \lambda_2, \lambda_1)_{\alpha_{23} \alpha_{13} \alpha_{12}}^{\beta_{23} \beta_{13} \beta_{12}} \beta_{12} R_{12}(\lambda_1, \lambda_2) \cdot \beta_{13} R_{13}(\lambda_1, \lambda_3) \cdot \beta_{23} R_{23}(\lambda_2, \lambda_3). \quad (8)
\end{aligned}$$

To simplify notation we will only write down the indices of the various spaces

$$R_{[12]} \cdot R_{[13]} \cdot R_{[23]} = S_{(12)(13)(23)} R_{[23]} \cdot R_{[13]} \cdot R_{[12]}, \quad (9)$$

$$R_{[23]} \cdot R_{[13]} \cdot R_{[12]} = \tilde{S}_{(23)(13)(12)} R_{[12]} \cdot R_{[13]} \cdot R_{[23]}. \quad (10)$$

Here the square brackets around the subscripts of  $R$  are to remind us that there are both external indices labeling the different  $R$  matrices and internal indices of Yang-Baxter type, while the brackets around the indices of  $S$  and  $\tilde{S}$  indicate that they are only external indices. Note that the order inside each bracketed pair is also important, and relabelings should be made with caution.

So far we have no relation between  $S$  and  $\tilde{S}$ , because they arose from different reversals. However, an application of these two reversals in succession yields the starting order, suggesting that the unitarity condition

$$\tilde{S}_{(23)(13)(12)} S_{(12)(13)(23)} = \delta_{(12)} \delta_{(13)} \delta_{(23)}, \quad (11)$$

should be satisfied, but again this is necessary only in the weak sense, i.e. when acting on a triple of matrices  $R$ .

In addition to the above we have to give a rule for exchanging  $R$ 's with disjoint indices. In general we could introduce a permutation operator  $Q$  by [20]

$$R_{[12]} R_{[34]} = Q_{(12)(34)} R_{[34]} R_{[12]}. \quad (12)$$

Since the internal (matrix) indices are disjoint it would be natural to choose  $Q = \delta\delta$ , but even if we later may take this conventional choice it is useful to carry along the operator  $Q$ , since it will turn out to be a good place-marker in the otherwise monotonous tetrahedron equation. Furthermore, Lawrence has proposed in [20] a variant of the tetrahedron equation where this  $Q$  operator is taken into account. [Among other things this allows for some additional (reductive) solutions of the form:  $Q_{(12)(34)}$  a solution of the quantum Yang-Baxter equation, while  $S_{(12)(13)(23)} = Q_{(12)(13)} Q_{(12)(23)} Q_{(13)(23)}$ .] In this paper we will only use the commutation and inversion properties:

$$Q_{(12)(34)} Q_{(13)(24)} = Q_{(13)(24)} Q_{(12)(34)}, \quad (13)$$

$$Q_{(12)(34)} Q_{(34)(12)} = \delta_{(12)} \delta_{(34)}, \quad (14)$$

$S$  and  $Q$  we defined by reversals of three and two  $R$ 's, respectively. When we considers ways of reversing more than three objects we get conditions for  $S$  (and  $Q$ ). In fact, because

of the dependence on *pairs* of indices we need to consider next the reversal of a product of six objects:  $R_{[ij]}$  where  $i \neq j \in \{1, 2, 3, 4\}$ . One particular case is the following:

$$\begin{array}{ccc}
\underbrace{R_{[12]} \cdot R_{[13]} \cdot R_{[23]}} \cdot R_{[14]} \cdot R_{[24]} \cdot R_{[34]} & & \\
\downarrow & S_{(12)(13)(23)} & \\
R_{[23]} \cdot R_{[13]} \cdot \underbrace{R_{[12]} \cdot R_{[14]} \cdot R_{[24]}} \cdot R_{[34]} & & \\
\downarrow & S_{(12)(14)(24)} & \\
R_{[23]} \cdot \underline{R_{[13]} \cdot R_{[24]}} \cdot R_{[14]} \cdot \underline{R_{[12]} \cdot R_{[34]}} & & \\
\Downarrow & Q_{(13)(24)} Q_{(12)(34)} & \\
R_{[23]} \cdot R_{[24]} \cdot \underbrace{R_{[13]} \cdot R_{[14]} \cdot R_{[34]}} \cdot R_{[12]} & & \\
\downarrow & S_{(13)(14)(34)} & \\
\underbrace{R_{[23]} \cdot R_{[24]} \cdot R_{[34]}} \cdot R_{[14]} \cdot R_{[13]} \cdot R_{[12]} & & \\
\downarrow & S_{(23)(24)(34)} & \\
R_{[34]} \cdot R_{[24]} \cdot \underline{R_{[23]} \cdot R_{[14]}} \cdot R_{[13]} \cdot R_{[12]} & & \\
\Downarrow & Q_{(23)(14)} & \\
R_{[34]} \cdot R_{[24]} \cdot R_{[14]} \cdot R_{[23]} \cdot R_{[13]} \cdot R_{[12]} & & 
\end{array}$$

Here the under-brace indicates which triple is reversed by the  $S$  and the underline means the terms are commuted using  $Q$ . At each step the multiplying obstruction matrix is written at the Dow-narrow, and they compose as

$$S_{(12)(13)(23)} S_{(12)(14)(24)} Q_{(13)(24)} Q_{(12)(34)} S_{(13)(14)(34)} S_{(23)(24)(34)} Q_{(23)(14)}.$$

There is precisely one other way to reverse the previous starting point:

$$\begin{array}{ccc}
R_{[12]} \cdot R_{[13]} \cdot \underline{R_{[23]} \cdot R_{[14]}} \cdot R_{[24]} \cdot R_{[34]} & & \\
\Downarrow & Q_{(23)(14)} & \\
R_{[12]} \cdot R_{[13]} \cdot R_{[14]} \cdot \underbrace{R_{[23]} \cdot R_{[24]} \cdot R_{[34]}} & & \\
\downarrow & S_{(23)(24)(34)} & \\
R_{[12]} \cdot \underbrace{R_{[13]} \cdot R_{[14]} \cdot R_{[34]}} \cdot R_{[24]} \cdot R_{[23]} & & \\
\downarrow & S_{(13)(14)(34)} & \\
\underline{R_{[12]} \cdot R_{[34]}} \cdot R_{[14]} \cdot \underline{R_{[13]} \cdot R_{[24]}} \cdot R_{[23]} & & \\
\Downarrow & Q_{(12)(34)} Q_{(13)(24)} & 
\end{array}$$



$$\begin{array}{ccc}
R_{[34]} \cdot \underbrace{R_{[12]} \cdot R_{[14]} \cdot R_{[24]}} & \downarrow & S_{(12)(14)(24)} \\
R_{[34]} \cdot R_{[24]} \cdot R_{[14]} \cdot \underbrace{R_{[12]} \cdot R_{[13]} \cdot R_{[23]}} & \downarrow & S_{(12)(13)(23)} \\
R_{[34]} \cdot R_{[24]} \cdot R_{[14]} \cdot R_{[23]} \cdot R_{[13]} \cdot R_{[12]} & & 
\end{array}$$

and for the last line the multiplier will be

$$Q_{(23)(14)} S_{(23)(24)(34)} S_{(13)(14)(34)} Q_{(12)(34)} Q_{(13)(24)} S_{(12)(14)(24)} S_{(12)(13)(23)}.$$

The equality of the above two expressions yields the tetrahedron equation:

$$\begin{aligned}
& S_{(12)(13)(23)} S_{(12)(14)(24)} Q_{(13)(24)} Q_{(12)(34)} S_{(13)(14)(34)} S_{(23)(24)(34)} Q_{(23)(14)} = \\
& Q_{(23)(14)} S_{(23)(24)(34)} S_{(13)(14)(34)} Q_{(12)(34)} Q_{(13)(24)} S_{(12)(14)(24)} S_{(12)(13)(23)}. \quad (15)
\end{aligned}$$

If  $Q = \delta\delta$  and we use the translation table 1=12, 2=13, 3=23, 4=14, 5=24, 6=34, we get the tetrahedron equation in the usual notation

$$S_{123} S_{145} S_{246} S_{356} = S_{356} S_{246} S_{145} S_{123}. \quad (16)$$

We note that the double index notation of (15) is more natural, because it identifies the points by the intersections of the two strings.

Let us finish this section by a comment on the spectral parameters. Each matrix  $S$  depends on three spectral parameters attached to each of the labels it carries. The derivation of this section gives a natural distribution of these parameters through the equation, as it does in the Yang-Baxter case. However, since we take the spectral parameter to be a projective vector in a fixed 3-dimensional complex space there is nonetheless a condition arising from the fact that four vectors in a three-dimensional space are necessarily dependent. This leads to the determinant condition given in [5] which, when expressed in terms of spherical angles associated with each of these vectors, is exactly the condition between the spectral parameters used in Zamolodchikov's construction of his solution, cf. [3]. An interesting question is whether this parametrization would correspond to the one that one would expect from the Baxterization procedure via the generalization of Coxeter groups underlying the tetrahedron equations, as proposed in [21, 22, 23]

## 2.3 The other tetrahedron equations

The main observation of this paper is that the above picture is incomplete in view of the fact that there are other starting points for the reversal of six  $R$ -matrices which will lead

to tetrahedron equations which in general are not equivalent to (15) or (16). In fact, we should investigate all possible starting configurations of matrices  $R$  and thus obtain a set of equations involving  $S$  as well as  $\tilde{S}$ .

It is not hard to find those starting configurations for which at least two triple reversals can be done. Without any loss of generality we may renumber the vector spaces and indices so that the first reversal is on  $\cdots R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdots$  resulting with  $\cdots R_{[23]} \cdot R_{[13]} \cdot R_{[12]} \cdots$  or in the reverse order:  $\cdots R_{[23]} \cdot R_{[13]} \cdot R_{[12]} \cdots$  resulting with  $\cdots R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdots$ . The next reversal must involve  $R_{[12]}$ ,  $R_{[23]}$  or  $R_{[13]}$ . In the first case the two other  $R$ 's that go with  $R_{[12]}$  must be on its right hand side, and can be numbered as  $R_{[14]} \cdot R_{[24]}$  or  $R_{[42]} \cdot R_{[41]}$  (note the order of indices, c.f. (9,10)), and the remaining term  $R_{[\{34\}]}$  (we do not yet know which index order works, this is reminded by the braces) can then be put in three different places resulting with six starting configurations:

$$\begin{array}{ll}
1 & R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdot R_{[14]} \cdot R_{[24]} \cdot R_{[\{34\}]} \\
1' & R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdot R_{[42]} \cdot R_{[41]} \cdot R_{[\{34\}]} \\
2' & R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdot R_{[\{34\}]} \cdot R_{[14]} \cdot R_{[24]} \\
2 & R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdot R_{[\{34\}]} \cdot R_{[42]} \cdot R_{[41]} \\
3 & R_{[\{34\}]} \cdot R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdot R_{[14]} \cdot R_{[24]} \\
3' & R_{[\{34\}]} \cdot R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdot R_{[42]} \cdot R_{[41]}
\end{array}$$

If the second reversal involves  $R_{[23]}$ , its (left hand side) companions can be numbered as  $R_{[42]} \cdot R_{[43]}$  or  $R_{[34]} \cdot R_{[24]}$ , and the remaining term  $R_{[\{14\}]}$  can again be distributed among the terms in three ways. This results with the following six possible starting configurations:

$$\begin{array}{ll}
4 & R_{[42]} \cdot R_{[43]} \cdot R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdot R_{[\{14\}]} \\
4' & R_{[34]} \cdot R_{[24]} \cdot R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdot R_{[\{14\}]} \\
5' & R_{[42]} \cdot R_{[43]} \cdot R_{[\{14\}]} \cdot R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \\
5 & R_{[34]} \cdot R_{[24]} \cdot R_{[\{14\}]} \cdot R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \\
6 & R_{[\{14\}]} \cdot R_{[42]} \cdot R_{[43]} \cdot R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \\
6' & R_{[\{14\}]} \cdot R_{[34]} \cdot R_{[24]} \cdot R_{[12]} \cdot R_{[13]} \cdot R_{[23]}
\end{array}$$

Finally, if the second reversal uses  $R_{[13]}$  we must put  $R_{[14]}$  on its left hand side and  $R_{[43]}$  of the right and  $R_{[\{24\}]}$  on either end, so that we can start with

$$\begin{array}{ll}
7 & R_{[\{24\}]} \cdot R_{[14]} \cdot R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdot R_{[43]} \\
8 & R_{[14]} \cdot R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdot R_{[43]} \cdot R_{[\{24\}]}
\end{array}$$

These starting points are then guaranteed to allow at least two triple reversals, but for a complete order reversal we have to do four triple reversals. It turns out that a third

reversal cannot be done in all cases, these bad cases are marked above with a prime. For each of the remaining eight starting points the first reversal can be done in precisely two ways, one of them is by  $S_{(12)(13)(23)}$ , the other one varying from case to case, these two alternatives give the two sides of the tetrahedron equations. We will not give the details here, the derivation follows the one done before and is easy to do since at each step there are no alternatives in applying the triple reversals.

The above classification can be repeated for starting points for which the first reversal is on  $\cdots R_{[23]} \cdot R_{[13]} \cdot R_{[12]} \cdots$  resulting with  $\cdots R_{[12]} \cdot R_{[13]} \cdot R_{[23]} \cdots$ . However, it turns out that these reversed starting points can be relabeled so that they give the same as the unreversed ones, except for cases 1 and 6. Thus we obtain, finally, eight different tetrahedron equations:

$$\begin{aligned}
1, 6 \quad & S_{(12)(13)(23)} S_{(12)(14)(24)} Q_{(13)(24)} Q_{(12)(34)} S_{(13)(14)(34)} S_{(23)(24)(34)} Q_{(23)(14)} = \\
& Q_{(23)(14)} S_{(23)(24)(34)} S_{(13)(14)(34)} Q_{(12)(34)} Q_{(13)(24)} S_{(12)(14)(24)} S_{(12)(13)(23)}, \\
2, 8r \quad & S_{(12)(13)(23)} Q_{(12)(43)} \tilde{S}_{(12)(42)(41)} \tilde{S}_{(13)(43)(41)} Q_{(23)(41)} Q_{(13)(42)} \tilde{S}_{(23)(43)(42)} = \\
& \tilde{S}_{(23)(43)(42)} Q_{(13)(42)} Q_{(23)(41)} \tilde{S}_{(13)(43)(41)} \tilde{S}_{(12)(42)(41)} Q_{(12)(43)} S_{(12)(13)(23)}, \\
3, 4r \quad & S_{(12)(13)(23)} S_{(12)(14)(24)} Q_{(13)(24)} \tilde{S}_{(43)(23)(24)} \tilde{S}_{(43)(13)(14)} Q_{(23)(14)} Q_{(43)(12)} = \\
& Q_{(43)(12)} Q_{(23)(14)} \tilde{S}_{(43)(13)(14)} \tilde{S}_{(43)(23)(24)} Q_{(13)(24)} S_{(12)(14)(24)} S_{(12)(13)(23)}, \\
4, 3r \quad & S_{(12)(13)(23)} S_{(42)(43)(23)} Q_{(42)(13)} \tilde{S}_{(42)(12)(14)} \tilde{S}_{(43)(13)(14)} Q_{(23)(14)} Q_{(43)(12)} = \\
& Q_{(43)(12)} Q_{(23)(14)} \tilde{S}_{(43)(13)(14)} \tilde{S}_{(42)(12)(14)} Q_{(42)(13)} S_{(42)(43)(23)} S_{(12)(13)(23)}, \\
5, 7r \quad & S_{(12)(13)(23)} Q_{(14)(23)} \tilde{S}_{(34)(24)(23)} \tilde{S}_{(34)(14)(13)} Q_{(24)(13)} Q_{(34)(12)} \tilde{S}_{(24)(14)(12)} = \\
& \tilde{S}_{(24)(14)(12)} Q_{(34)(12)} Q_{(24)(13)} \tilde{S}_{(34)(14)(13)} \tilde{S}_{(34)(24)(23)} Q_{(14)(23)} S_{(12)(13)(23)}, \\
1r, 6r \quad & \tilde{S}_{(23)(13)(12)} \tilde{S}_{(24)(14)(12)} Q_{(24)(13)} Q_{(34)(12)} \tilde{S}_{(34)(14)(13)} \tilde{S}_{(34)(24)(23)} Q_{(14)(23)} = \\
& Q_{(14)(23)} \tilde{S}_{(34)(24)(23)} \tilde{S}_{(34)(14)(13)} Q_{(34)(12)} Q_{(24)(13)} \tilde{S}_{(24)(14)(12)} \tilde{S}_{(23)(13)(12)}, \\
7, 5r \quad & S_{(12)(13)(23)} Q_{(14)(23)} Q_{(12)(43)} S_{(14)(13)(43)} S_{(24)(23)(43)} Q_{(24)(13)} \tilde{S}_{(24)(14)(12)} = \\
& \tilde{S}_{(24)(14)(12)} Q_{(24)(13)} S_{(24)(23)(43)} S_{(14)(13)(43)} Q_{(12)(43)} Q_{(14)(23)} S_{(12)(13)(23)}, \\
8, 2r \quad & S_{(12)(13)(23)} Q_{(14)(23)} Q_{(12)(43)} S_{(14)(13)(43)} S_{(14)(12)(42)} Q_{(13)(42)} \tilde{S}_{(23)(43)(42)} = \\
& \tilde{S}_{(23)(43)(42)} Q_{(13)(42)} S_{(14)(12)(42)} S_{(14)(13)(43)} Q_{(12)(43)} Q_{(14)(23)} S_{(12)(13)(23)}.
\end{aligned} \tag{17}$$

### 3 Interpretation

To get a better understanding of the equations (17), let us look at the simplified case where the matrices  $Q$  are all taken equal to one, and investigate the geometric meaning of the set of equations we have obtained.

### 3.1 Reduction under unitarity

Let us first renumber the indices in (17) so that inside each bracket  $(ij)$  we have  $i < j$ . This is accomplished by the following cyclic renumberings: 1 none, 2 (1234), 3 (34), 4 (234), 5 none, 1r none, 7 (34), 8 (234). After this it turns out that in each  $S$  the indices are  $S_{(ij)(ik)(jk)}$  with  $i < j < k$ , and if  $i, j, k, l$  is a permutation of  $\{1, 2, 3, 4\}$  we can use the shorthand notation  $S_l := S_{(ij)(ik)(jk)}$ , similarly  $\tilde{S}_l := \tilde{S}_{(jk)(ik)(ij)}$ . For  $Q$ 's we use  $Q_i := Q_{(1i)(jk)}$ , where  $j < k$  and  $i, j, k$  is a permutation of  $\{2, 3, 4\}$  (recall also Eqn. (14)). After multiplying each line from left and right by suitable  $\tilde{S}^{-1}$  and  $Q$  to eliminate all  $\tilde{S}$ 's and  $Q^{-1}$ 's, exchanging left and right hand sides in equations 3, 5 and 8, and writing the whole set in a different order yields

$$\begin{aligned}
1, 6 & \quad Q_4 \ S_4 \ S_3 \ Q_3 Q_2 \ S_2 \ S_1 = S_1 \ S_2 \ Q_2 Q_3 \ S_3 \ S_4 \ Q_4, \\
7, 5r & \quad Q_4 \tilde{S}_4^{-1} \ S_3 \ Q_3 Q_2 \ S_2 \ S_1 = S_1 \ S_2 \ Q_2 Q_3 \ S_3 \ \tilde{S}_4^{-1} Q_4, \\
4, 3r & \quad Q_4 \tilde{S}_4^{-1} \tilde{S}_3^{-1} Q_3 Q_2 \ S_2 \ S_1 = S_1 \ S_2 \ Q_2 Q_3 \ \tilde{S}_3^{-1} \tilde{S}_4^{-1} Q_4, \\
2, 8r & \quad Q_4 \tilde{S}_4^{-1} \tilde{S}_3^{-1} Q_3 Q_2 \tilde{S}_2^{-1} \ S_1 = S_1 \ \tilde{S}_2^{-1} Q_2 Q_3 \ \tilde{S}_3^{-1} \tilde{S}_4^{-1} Q_4, \\
1r, 6r & \quad Q_4 \tilde{S}_4^{-1} \tilde{S}_3^{-1} Q_3 Q_2 \tilde{S}_2^{-1} \tilde{S}_1^{-1} = \tilde{S}_1^{-1} \tilde{S}_2^{-1} Q_2 Q_3 \ \tilde{S}_3^{-1} \tilde{S}_4^{-1} Q_4, \\
5, 7r & \quad Q_4 \ S_4 \ \tilde{S}_3^{-1} Q_3 Q_2 \tilde{S}_2^{-1} \tilde{S}_1^{-1} = \tilde{S}_1^{-1} \tilde{S}_2^{-1} Q_2 Q_3 \ \tilde{S}_3^{-1} \ S_4 \ Q_4, \\
3, 4r & \quad Q_4 \ S_4 \ S_3 \ Q_3 Q_2 \tilde{S}_2^{-1} \tilde{S}_1^{-1} = \tilde{S}_1^{-1} \tilde{S}_2^{-1} Q_2 Q_3 \ S_3 \ S_4 \ Q_4, \\
8, 2r & \quad Q_4 \ S_4 \ S_3 \ Q_3 Q_2 \ S_2 \ \tilde{S}_1^{-1} = \tilde{S}_1^{-1} \ S_2 \ Q_2 Q_3 \ S_3 \ S_4 \ Q_4.
\end{aligned} \tag{18}$$

This is the final form of the equations, when considered from the algebraic point of view. Clearly, if the unitarity condition (11) holds we have  $\tilde{S}_i^{-1} = S_i$  and all equations are identical.

Equations of exactly the same form but with  $Q = \delta\delta$  were presented in [18], where they were derived in quite a different context: The study in [18] was based on bicategories and the equations were presented as a theorem stating that certain bicategories will satisfy this set.

### 3.2 Geometric interpretation

Above we have given an algebraic derivation, but the tetrahedron equations can also be derived by other approaches, for example by the geometric approach of straight string scattering, which we will consider next. In this approach the unitarity condition does not arise, and we still have the ordering problem discussed in the introduction. (In the following we will drop the  $Q$ -matrices.)

In order to have a geometric interpretation of (17) we renumber the indices so that in

each equation the indices of an  $S$ -matrix contain the same set of numbers as in Case 1. The required renumberings are: 1,6 none, 2,8r none, 3,4r (1324) and exchange of left and right hand sides, 4,3r (132), 5,7r (1234) and exchange, 1r,6r none, 7,5r (123), 8,2r (14) and exchange, this yields

$$\begin{aligned}
1, 6 & S_{(12)(13)(23)} S_{(12)(14)(24)} S_{(13)(14)(34)} S_{(23)(24)(34)} = S_{(23)(24)(34)} S_{(13)(14)(34)} S_{(12)(14)(24)} S_{(12)(13)(23)} \\
2, 8r & S_{(12)(13)(23)} \tilde{S}_{(12)(42)(41)} \tilde{S}_{(13)(43)(41)} \tilde{S}_{(23)(43)(42)} = \tilde{S}_{(23)(43)(42)} \tilde{S}_{(13)(43)(41)} \tilde{S}_{(12)(42)(41)} S_{(12)(13)(23)} \\
3, 4r & \tilde{S}_{(12)(32)(31)} \tilde{S}_{(12)(42)(41)} S_{(34)(31)(41)} S_{(34)(32)(42)} = S_{(34)(32)(42)} S_{(34)(31)(41)} \tilde{S}_{(12)(42)(41)} \tilde{S}_{(12)(32)(31)} \\
4, 3r & S_{(31)(32)(12)} S_{(41)(42)(12)} \tilde{S}_{(41)(31)(34)} \tilde{S}_{(42)(32)(34)} = \tilde{S}_{(42)(32)(34)} \tilde{S}_{(41)(31)(34)} S_{(41)(42)(12)} S_{(31)(32)(12)} \\
5, 7r & \tilde{S}_{(31)(21)(23)} \tilde{S}_{(41)(21)(24)} \tilde{S}_{(41)(31)(34)} S_{(23)(24)(34)} = S_{(23)(24)(34)} \tilde{S}_{(41)(31)(34)} \tilde{S}_{(41)(21)(24)} \tilde{S}_{(31)(21)(23)} \\
1r, 6r & \tilde{S}_{(23)(13)(12)} \tilde{S}_{(24)(14)(12)} \tilde{S}_{(34)(14)(13)} \tilde{S}_{(34)(24)(23)} = \tilde{S}_{(34)(24)(23)} \tilde{S}_{(34)(14)(13)} \tilde{S}_{(24)(14)(12)} \tilde{S}_{(23)(13)(12)} \\
7, 5r & S_{(23)(21)(31)} S_{(24)(21)(41)} S_{(34)(31)(41)} \tilde{S}_{(34)(24)(23)} = \tilde{S}_{(34)(24)(23)} S_{(34)(31)(41)} S_{(24)(21)(41)} S_{(23)(21)(31)} \\
8, 2r & \tilde{S}_{(23)(13)(12)} S_{(41)(42)(12)} S_{(41)(43)(13)} S_{(42)(43)(23)} = S_{(42)(43)(23)} S_{(41)(43)(13)} S_{(41)(42)(12)} \tilde{S}_{(23)(13)(12)}
\end{aligned} \tag{19}$$

These orderings were derived algebraically but one can give a geometric rule that produces the same:

**Geometric rule for label ordering:** Draw a line on the plane, outside the region of string intersections. When the line moves, without changing its direction, it will sweep across the intersection region. *For each scattering matrix write the indices of the corresponding triangle corners in the order the line hits them.* If the order is counterclockwise, use  $\tilde{S}$ .

In Figure 3 we have redrawn Figure 1 with 8 approaching lines. These sweeping lines give exactly the orderings that were obtained by the algebraic method after relabeling (19).

### 3.3 Connection with Bruhat order $B(4,2)$

The above concrete geometrical approach can be made more precise using the notion of higher Bruhat orders, introduced by Manin and Shechtman in [15, 16]. This provides the proper algebraic setting for the description of the general  $d$ -simplex equations, viewed as higher-order intertwining or braiding objects. The setting is that of moves of hyperplanes embedded in a higher-dimensional space. Realizations can be constructed in terms of generators of the fundamental group of configuration spaces formed by the complements of such configurations [24]. An explicit realization was constructed by Lawrence in [25].

To make our account self-contained, we will briefly describe the Manin-Shechtman construction, cf. [15, 16]. (An alternative description was given recently in [26].) First,

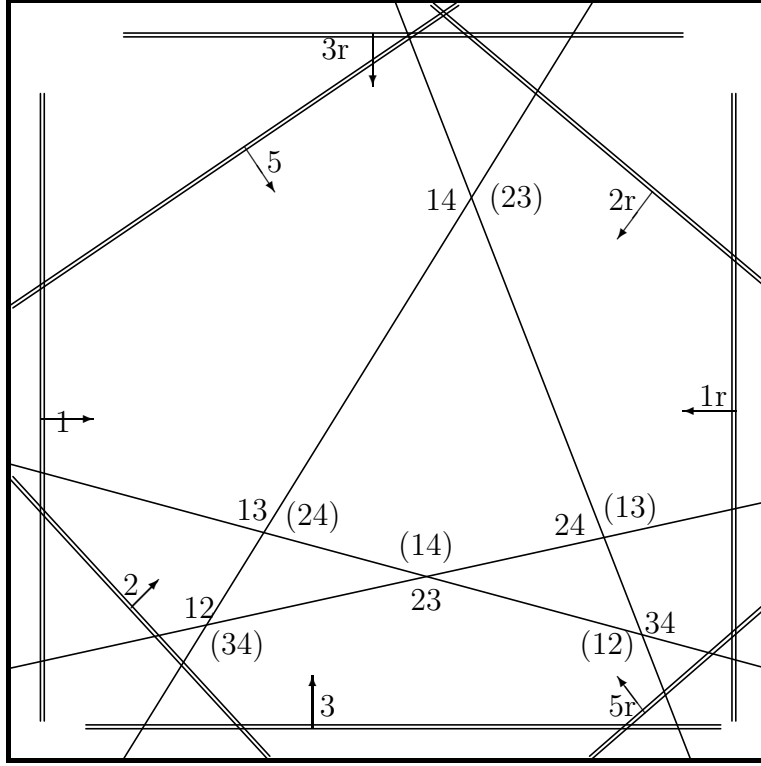


Figure 3: How the possible algebraic orderings can be derived geometrically. The vertices are ordered according to the order in which the moving (double) lines hit them. The complementary numbering given in parentheses is used with the Bruhat ordering of Sec. 3.3

for any pair of integers,  $n, k$ , with  $n \geq k \geq 1$ , they introduce the set of  $k$ -element subsets  $C(n, k)$  of the set  $\underline{n} = \{1, 2, \dots, n\}$ , whose elements will be denoted by  $(i_1 i_2 \dots i_k)$  in increasing order,  $i_1 < i_2 < \dots < i_k$ . For any given element  $c \in C(n, k)$ , we denote by  $\hat{c}_j$  the element of  $C(n, k-1)$  obtained by removing the  $j$ th element  $i_j$ , ( $1 \leq j \leq k$ ), from the tuple  $c$ .

Next, we consider the set of total orders on the set  $C(n, k)$ . For this purpose we need to select from  $C(n, k)$  only those orderings that descend from  $C(n, k+1)$ , i.e. the elements  $\hat{d}_j \in C(n, k)$ , ( $j = 1, \dots, k+1$ ), coming from *each*  $d \in C(n, k+1)$  by applying the  $\hat{\cdot}$ -operation described above. They are ordered in either ascending or in descending order,

$$\hat{d}_1 < \hat{d}_2 < \dots < \hat{d}_{k+1} \quad \text{or} \quad \hat{d}_1 > \hat{d}_2 > \dots > \hat{d}_{k+1} . \quad (20)$$

The set of all such total orderings is called  $A(n, k)$ , and its elements can be written as chains  $a = c_1 c_2 \dots c_n$ ,  $c_i \in C(n, k)$ , with  $c_1 < c_2 < \dots < c_n$  in the given ordering by  $a$ .

**Example:** Consider the set  $A(4, 2)$  which are constructed according to the scheme above. First we need the sets  $C(4, 2)$  having six elements:

$$C(4, 2) = \{(12), (13), (14), (23), (24), (34)\} =: \{c_1, c_2, c_3, c_4, c_5, c_6\},$$

and  $C(4, 3)$  having four elements:

$$C(4, 3) = \{(123), (124), (134), (234)\}.$$

From the latter set we can construct the elements  $\hat{d}_j$  for each  $d \in C(4, 3)$ , leading to the following list of conditions on the orderings:

$$\begin{aligned} (23) < (13) < (12) & \quad \text{or} \quad (23) > (13) > (12), \\ (24) < (14) < (12) & \quad \text{or} \quad (24) > (14) > (12), \\ (34) < (14) < (13) & \quad \text{or} \quad (34) > (14) > (13), \\ (34) < (24) < (23) & \quad \text{or} \quad (34) > (24) > (23). \end{aligned}$$

A geometric picture is very useful to find out which orderings on  $C(4, 2)$  (i.e., which combinations of the above list) are actually allowed. It turns out that the allowed orderings are exactly those that can be obtained from figure 3, by looking at it from various directions. At this point we should keep also those orderings that differ only by an exchange of elements not directly connected, e.g, by a small tilt on direction 3 we can have  $c_4c_2c_5 \dots$  and  $c_4c_5c_2 \dots$ . In this way we obtain the set  $A(4, 2)$ , containing e.g.

$$A(4, 2) = \{c_1c_2c_3c_4c_5c_6, c_4c_2c_5c_3c_1c_6, c_6c_1c_3c_5c_2c_4, \dots\}.$$

To obtain the Bruhat orders  $B(n, k)$ , we need to consider the set  $A(n, k)$  up to an inversion, i.e. selecting only one of each possibility in (20). So, the set  $Inv(a)$  of inversions on an element  $a \in A(n, k)$  is a subset  $d \in C(n, k+1)$  such that  $\hat{d}_1 < \hat{d}_2 < \dots < \hat{d}_{k+1}$ . Furthermore, we need to introduce an equivalence under adjacency. Two elements in  $A(n, k)$  will be called equivalent,  $a \sim a'$ , provided  $a'$  is obtained from  $a = c_1c_2 \dots c_n$  by the permutation of two adjacent subsets  $c_j$  and  $c_{j+1}$ , containing in the union at least  $k+2$  elements. The Bruhat orders are then the equivalence classes in  $A(n, k)$  under this equivalence relation, i.e. they are contained in the set  $B(n, k) = A(n, k)/\sim$ .

Let us now see what this amounts to in the case of  $A(4, 2)$ . The adjacent elements in  $C(4, 2)$  are exactly the ones that have no entries in common. In this way they correspond to the orderings up to interchanging the subsets (12) and (34), (13) and (24), and (14)

and (23). Thus, with the above identification of orders, we get for  $B(n, k)$ :

$$B(4, 2) = \{[c_1 c_2 c_3 c_4 c_5 c_6], [c_6 c_5 c_4 c_3 c_2 c_1], [c_4 c_2 c_1 c_3 c_5 c_6], [c_1 c_2 c_3 c_6 c_5 c_4], \\ [c_4 c_5 c_6 c_3 c_2 c_1], [c_6 c_5 c_3 c_1 c_2 c_4], [c_6 c_1 c_3 c_2 c_5 c_4], [c_4 c_5 c_2 c_3 c_1 c_6]\} , \quad (21)$$

corresponding to directions  $1r, 1, 2r, 5r, 5, 2, 3, 3r$  in Fig. 3.

It is easily noted that this partial ordering when imposed on  $B(4, 2)$  corresponds exactly to the configurations in the obstruction derivation of the eight tetrahedron equations. At this point we can note the connection with the work on braided monoidal 2-categories, cf. [19], that also leads to the set of eight tetrahedron equations, albeit from quite a different point of view.

## 4 Conclusions

The statement of this paper is the following: what is usually referred to as the tetrahedron equation is actually one of a system of eight coupled equations that can be derived systematically from the collection of all consistency conditions that arise from the underlying set of trilinear equations (9) and (10). We have shown also that the various classes of starting configurations that lead to these different equations are labeled by a new algebraic object, which is the higher Bruhat order  $B(4, 2)$  introduced by Manin and Shechtman. It is obvious that these considerations can, in principle, be extended to any dimension, i.e. to obtain systems of  $D$ -simplex equations for any  $D = 2, 3, \dots$ .

Of course, our derivation comes down to the same type of combinatorics that is behind the description in terms of 2-category theory, [19], but ours is closer to the physical interpretation. Furthermore, we hope that the obstruction derivation might eventually lead to the derivation of solutions to the system of eight equations (in the cases that the set cannot be reduced to one single equation) along the same lines as the derivation of solutions in the papers by Korepanov, [10, 11]. We have investigated the known solutions of the tetrahedron equations in [10, 12], but unfortunately these all fall into the class of unitary solutions for which the system collapses (that is, the second equation of (18) did not have other solutions than those with  $\tilde{S}_4^{-1} = S_4$ ). However, it cannot be ruled out that nontrivial solutions of the full non-degenerate system (18) exist even though it might not be so easy to find such solutions.



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